Correlation dimension of intermittent signals

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We investigate the breaking of proper self-similarity of attractors in the presence of intermittency. We show that this can lead to dramatically too small values of the numerically estimated correlation dimension D_2, which we relate, in the case of type I intermittency, to universal scaling properties in the vicinity of the critical value of the control parameter. For spatially extended systems we study the influences of space-time intermittency on the correlation dimension. [S1063-651X(97)01307-X]

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The estimation of invariant quantities like dimensions, entropies, or Lyapunov exponents has become quite a ubiquitous task in the analysis of chaotic time series. It is impossible to numerically perform the limits given in the mathematical definition of these quantities, since the attractors formed by time series are usually represented by a finite number of points and experimental observables are additionally corrupted by noise. In the case of ''well behaving'' low dimensional systems this is not a severe problem at all. There one can estimate, e.g., the correlation dimension even on relatively large length scales with good accuracy. By ''well behaving'' we mean that the structure of the attractor is more or less independent of the length scales we look at (or in other words, the self-similarity is visible on large scales, already). But this may not always be the case. The properties might differ quite strongly with respect to the length scales. A trivial example of this length scale dependency is a deterministic system, which is corrupted by noise. In this case the true dimension is infinity, but on large scales one might see (if the noise is not too large) the dimension of the noise free attractor.

Another example is the one we want to investigate in this paper in more detail. The system we are looking at is supposed to show intermittent behavior. That means, the time evolution of the system looks periodic for a certain time, but occasionally shows chaotic bursts.

One possible mechanism for this behavior (the so called type I intermittency [1]) is illustrated in Fig. 1. There we show the graph of the third iterate of the logistic map [$f(x)=1-ax^2$] for $a=1.7425$ (left plot). For $a=a_c=7/4$ the map undergoes a bifurcation from chaotic behavior $a<a_c$ to a stable period 3 orbit $a>a_c$. Thus, for the parameter value of a shown in the figure, the map is chaotic. But one can see that close to the values of x where the fixed points arise for $a=a_c$ the map is already near the $y=x$ line. That means that once the trajectory of $f\circ f\circ f$ comes close to these points, it stays there for a certain time (right plot in Fig. 1), and a time series of f looks roughly periodic. It is known that the average length $\langle l\rangle$ of the almost periodic segments scales like

$$\langle l\rangle\sim(a-a_c)^{1/2} \tag{1}$$

for type I intermittency [2,3]. As we will see in the following, this behavior has dramatic consequences for the correlation integral.

As another example for a low dimensional system, we treat the Lorenz system [4]. It has a periodic solution for the parameter values $b=8/3$, $\sigma=10$, and $r\approx166$ [2]. For these parameters we will see a one dimensional, nonchaotic limit cycle. Increasing r a bit will lead to intermittent behavior [1]. A delay plot of the x coordinate for $r=166.07$ is shown in Fig. 2. Most of the points are close to the periodic orbit, whereas the remainder of the attractor is poorly sampled by the short chaotic bursts.

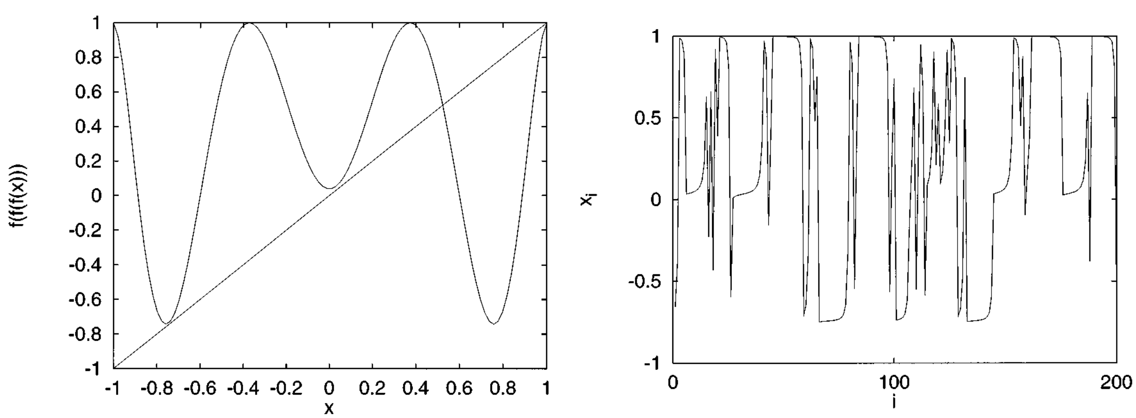

FIG. 1. Third iterate of the logistic map with a parameter a close to the bifurcation to the period 3 orbit (left panel) and the resulting time series (right panel).

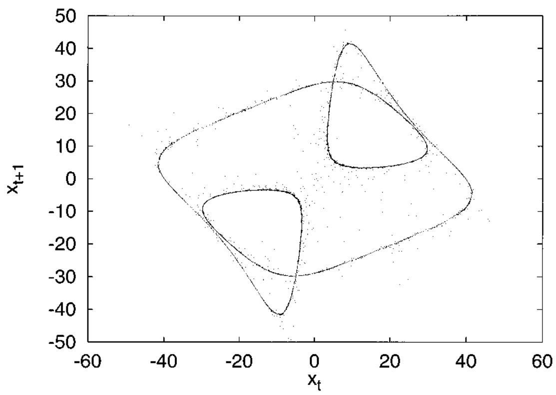

FIG. 2. Delay plot of the x coordinate of the Lorenz system with strong intermittency. The parameters were: $b=8/3$, $\sigma=10$, $r=166.07$.

Figure 3 shows the results for the correlation dimension estimate of these data. On large length scales the dimension of the system is close to 1. Only on much smaller length scales do we find the crossover to the correct dimension (>2).

This behavior can be explained by studying the properties of the correlation dimension. The correlation integral is defined by [5]

$$C_2(\epsilon,m)=\frac{1}{N(N-1)}\sum_{c=1}^{N}\sum_{n\neq c}\Theta(\epsilon-\|\vec{x}_c-\vec{x}_n\|),\qquad(2)$$

where m is the embedding dimension, ϵ the length scale, and N the number of vectors $\vec{x}_i=(x_i,x_{i-1},\ldots,x_{i-m+1})$. Now we can divide the first sum on the right-hand side into two parts: one containing the centers inside the laminar phase, the other one containing centers inside the chaotic bursts. This gives

$$C_2(\epsilon,m)=p_{\rm in}C_2^{\rm in}(\epsilon,m)+p_{\rm out}C_2^{\rm out}(\epsilon,m),\qquad(3)$$

with $p_{\rm in(out)}=N_{\rm in(out)}/N$ being the fraction of centers lying in the laminar and the chaotic phase, respectively. From this correlation integral one gets the correlation dimension by

$$D_2(\epsilon,m)=\frac{\partial \ln C_2(\epsilon,m)}{\partial \ln\epsilon}.\qquad(4)$$

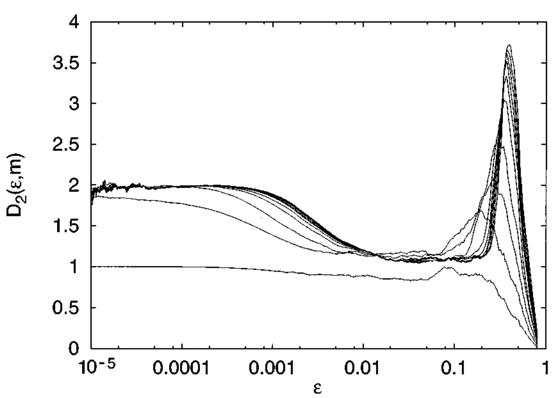

FIG. 3. Results for the estimate of the correlation dimension for the data shown in Fig. 2. The length of the time series was 10^6.

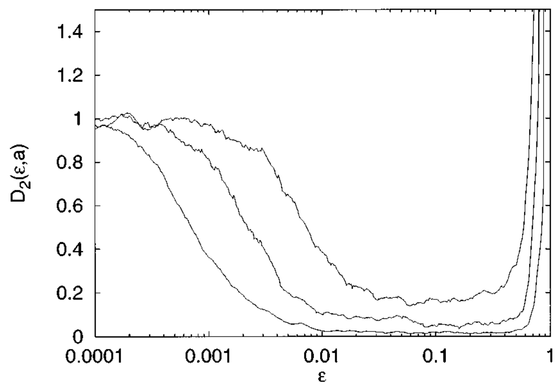

FIG. 4. Shown are the dimension estimates of the logistic map for three values of a ($a=1.7499, 1.749\,99, 1.749\,999$ from above to below) close to a_c. The embedding dimension was set to $m=5$ for all a.

Using Eq. (3) one easily gets

$$D_2(\epsilon,m)=p'_{\rm in}D_2^{\rm in}(\epsilon,m)+p'_{\rm out}D_2^{\rm out}(\epsilon,m),\qquad(5)$$

with

$$p'_{\rm in(out)}=\frac{\sum_{\rm in(out)}\sum_{n}\Theta(\epsilon-\|\vec{x}_{\rm in(out)}-\vec{x}_n\|)}{\sum_{c}\sum_{n}\Theta(\epsilon-\|\vec{x}_c-\vec{x}_n\|)},$$

which is the fraction of pairs found in the laminar phase and the chaotic bursts, respectively.

On large length scales the "partial" dimension $D_2^{\rm in}(\epsilon,m)$ is close to 1, while $D_2^{\rm out}(\epsilon,m)$ is close to 2. (Of course, on length scales comparable to the attractor size, the correlation integral is mainly dominated by folding effects and $D_2(\epsilon,m)$ may be much larger than 2.) Furthermore, most of the points (see Fig. 2) are located in the laminar phase, which means that $p'_{\rm in}$ is large. This explains the behavior of the correlation dimension on large length scales ϵ. If we decrease ϵ the "fine structure" of the laminar phase is resolved. That means, it becomes visible that it is not a limit cycle, but only close to one and $D_2^{\rm in}(\epsilon,m)$ increases, so that we eventually see the global dimension of the attractor.

What can we say about the behavior of $D_2(\epsilon)$ for values of ϵ where the laminar parts still look like a periodic solution? Let $\langle l\rangle$ be the average length of the laminar parts and $\langle t\rangle$ the average length of the chaotic bursts. Then, the probability of a point lying in the laminar phase is given by

$$p_{\rm in}=\frac{\langle l\rangle}{\langle l\rangle+\langle t\rangle}.\qquad(6)$$

From Eq. (1) we know that $\langle l\rangle$ scales like $1/\sqrt{a_c-a}$. On the other hand, $\langle t\rangle$ does not depend on a_c-a (at least not strongly), since it is determined by the global properties of the map, which do not change (strongly) under small changes of a. Thus, for small variations of a, we can expect $\langle t\rangle$ being a constant. With this we can conclude that the large scale dimension behaves like

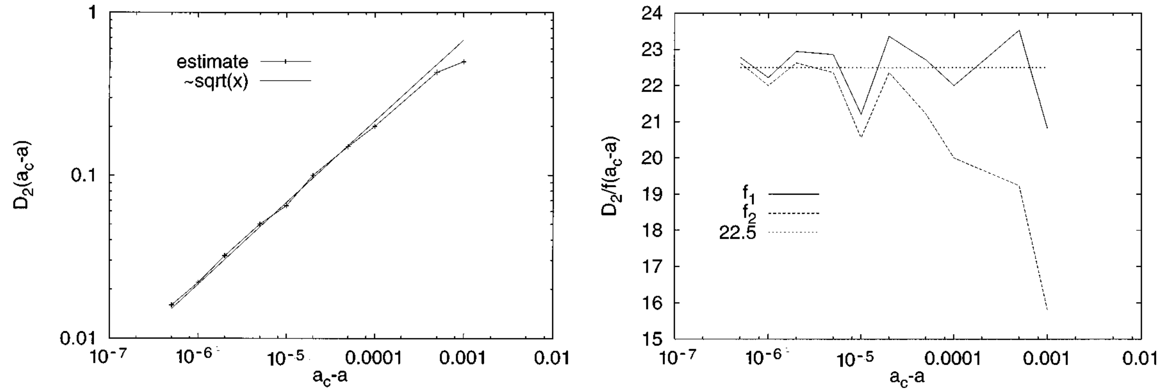

FIG. 5. Estimated large scale dimension as a function of $a_c - a$. The left plot shows the scaling of D_2. The right plot shows the behavior of $D_2/f(a_c-a)$, where f is $f_1 = \sqrt{a_c - a}$ and $f_2 = \sqrt{a_c - a}\Psi(a)$, respectively.

$$D_2(\epsilon,m) = \frac{\langle l\rangle}{\langle l\rangle + \langle t\rangle} D_2^{\mathrm{in}}(\epsilon,m) + \frac{\langle t\rangle}{\langle l\rangle + \langle t\rangle} D_2^{\mathrm{out}}(\epsilon,m). \quad (7)$$

(For the dimension we have to use p'_{in} instead of p_{in}. But the scaling law is the same for both quantities, if the neighborhoods of the points in the laminar and the chaotic parts of the phase space do not overlap too strongly.)

We want to check the scaling behavior Eq. (7) on the logistic map $f(x) = 1 - ax^2$. There we find the intermittent behavior close to $a_c = 7/4$ [3], where the period 3 window occurs. For the case of the logistic map we have $D_2^{\mathrm{in}}(\epsilon,m) = 0$ and $D_2^{\mathrm{out}}(\epsilon,m) = 1$, and Eq. (7) reduces to

$$D_2(\epsilon,m) \approx \sqrt{a_c - a}\,\Psi(a), \quad (8)$$

where

$$\Psi(a) = \frac{1}{1 + \langle t\rangle/\langle l\rangle} = \frac{1}{1 + \mathrm{const}\sqrt{a_c - a}}$$

gives the corrections to the square-root behavior and can easily be calculated from Eq. (7).

Figure 4 shows the estimate of the dimensions on large length scales for different values of a. The closer a is to a_c, the smaller is the minimum of the estimated dimension.

In Fig. 5 (left plot) we show the estimated dimension on large scales as a function of $a_c - a$. The plot shows that the dimensions scale pretty well like the expected square root. The right plot shows $D_2(\epsilon,m)/f(a_c - a)$, with f being $f_1 = \sqrt{a_c - a}$ and $f_2 = \sqrt{a_c - a}\Psi(a)$, respectively, which is more sensitive to using the correct scaling law, and we see that Eq. (8) describes the results correctly. The constant in Ψ was fitted to be 7.

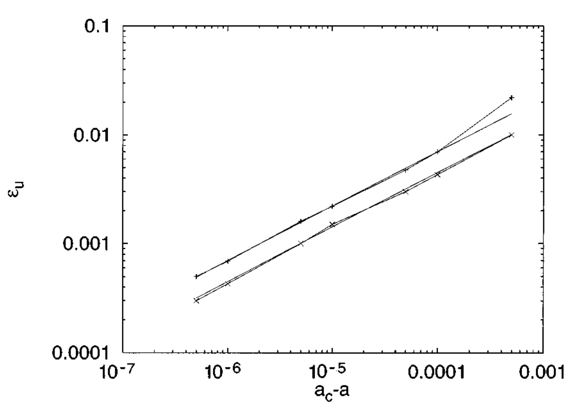

FIG. 6. ϵ_u as a function of $a - a_c$ for two different values of t [$t = 0.5$ (upper curve) and $t = 0.7$ (lower curve)]. Additionally the square roots are plotted.

Also for the length scales, where the crossover of the correlation dimension takes place, a scaling law should hold. We can define a characteristic length scale of the laminar phases. The average ''time'' the laminar phases last is given by $\langle l\rangle$ and the velocity with which the trajectory ''travels'' through the laminar part is in first approximation proportional to $a - a_c$. This defines a length scale

$$\epsilon_c \propto (a - a_c)\langle l\rangle \propto \sqrt{a - a_c}. \quad (9)$$

It is very difficult to get this ϵ_c directly from data of the dimension estimate. Instead, we show in Fig. 6 the values of a length scale ϵ_u, which is defined as the length scale, where the correlation dimension crosses a given threshold u. In Fig. 6 we show ϵ_u as a function of $a - a_c$ for two different values of u, namely, $u = 0.5$ and $u = 0.7$, and one can see that the scaling law (9) holds pretty well.

In spatially extended systems one often observes another kind of intermittency, which is called space-time intermittency [6,7]. Space-time intermittency means that we have intermittent behavior in time as well as in space. This leads to a very complex and inhomogeneous structure of the system.

As for low dimensional systems we must expect that the estimates of the correlation dimension of this kind of intermittent systems might show similar ''misleading'' results, as they do for low dimensional systems.

As an example of a spatially extended system, we study a coupled map lattice (CML). This is a system of N maps spatially arranged as a linear chain (in the one dimensional case), which are usually coupled by a nearest neighbor mechanism

$$x_i^{t+1} = (1 - 2\sigma) f(x_i^t) + \sigma[f(x_{i+1}^t) + f(x_{i-1}^t)], \quad (10)$$

where $i = 1, \ldots, N$ is the spatial index of the maps, t the (discrete) time, and f a chaotic map. Here f is the logistic

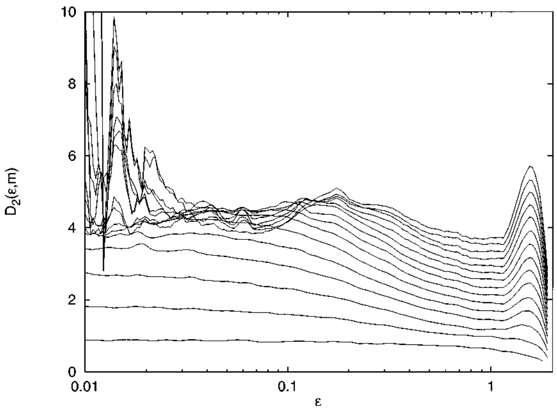

FIG. 7. Dimension estimate of a system of 100 coupled Ulam maps based on a scalar time series. The coupling constant was set to $\sigma=1/3$. The embedding dimensions range from $m=1$ to $m=15$.

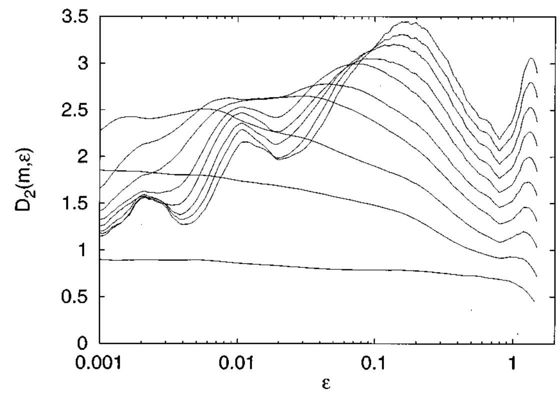

FIG. 8. Similar to Fig. 7, but for $\sigma=0.16$ and $m=1,\dots,10$.

map $f(x)=1-ax^2$ in the regime of fully developed chaos $(a=2)$. $\sigma\in[0,1/2]$ is the coupling constant. Compared to the more physical partial differential equations, the advantage of this system is that it is discrete in space and time and thus very easy to iterate. Additionally, the Jacobian of this system is easy to calculate and one can estimate the Lyapunov exponents to whatever precision (in a numerical sense) one wants. Using the Pesin identity and the Kaplan-Yorke formula one easily obtains the entropy and the dimension of the system, so that we can check the results from the dimension estimate based on the time series analysis directly with the results obtained by the Lyapunov exponents. Of course, the time series analysis is not able to yield the full dimension of the system. The reason is that for a typical system size $N=100$ and a coupling constant $\sigma=1/3$ (which is sometimes called the democratic coupling) the Kaplan-Yorke formula yields a dimension of $D\approx 55$. Even if we assume that the system is multifractal, so that the correlation dimension is smaller than the information dimension, it is impossible to reach such values using the time series analysis [8,9].

Figure 7 shows a typical plot of the correlation dimension estimate based on a scalar time series $\{x_i^t\}_{t=1,\dots,T}$ for the above mentioned parameters. For our treatments the interesting part of the plot is the breakdown of the estimate for $\epsilon\leqslant 0.2$. This breakdown shows a signature similar to the one observed in the dimension estimate for systems with strong linear correlations [10]. We will see that the effects shown in Fig. 7 are of completely different origin.

The behavior in our system is determined by a kind of space-time intermittency. If one observes the time evolution of the system one sees that the spatial degrees of freedom behave most of the time irregularly. But occasionally, this irregularity is replaced by laminar windows occurring temporarily in parts of the system. To study the structure of these laminar parts we reduced the system size. If the system size is small enough, the whole system eventually ''relaxes'' to the laminar solution and one can investigate its properties in detail. For our special parameter $(\sigma=1/3)$ this is very difficult. Depending on the system size the laminar parts are periodic in time with either period 2 or period 4 or even quasiperiodic. Also the spatial structure of the (quasi-) periodic solutions is ambiguous. We found period 7 in space, but also a structure which shows no clear periodicity. This complicated behavior also appears in the large system. Depending on the size of the laminar phases in the large system, one can find any of the mentioned solutions. Thus one cannot relate the properties of the correlation dimension to a single periodic solution. To study its behavior in more detail, let us treat another value of σ, namely, $\sigma=0.16$. For this coupling the breakdown is even more pronounced (see Fig. 8).

For this value of σ there exists a unique periodic solution [11], which has period 2 in time and period 4 in space with the spatial structure: $\dots xxyyxxyy \dots$, where x and y are

$$y(x)=\frac{1}{4(1-2\sigma)}\left(1\pm\left[9-\frac{4(1-\sigma)^2}{(1-2\sigma)^2}\right]^{1/2}\right). \qquad (11)$$

Linear stability analysis shows, that this solution is stable in the interval $\sigma\in[0.14{:}0.19]$ for all $N=4n$ with $n\in\mathbf{N}$ and periodic boundary conditions. Thus, the chaotic behavior we observe in this case is only transient and represents a chaotic repeller.

Independent of N, parts of the system are temporarily in the laminar (periodic) phase. Of course, these periodic subsystems are disturbed by the rest of the systems and, therefore, have finite life times only. This is one mechanism for space-time intermittency and we can expect some behavior for the correlation integral similar to what we found for the low dimensional intermittent systems.

The difference in the dimension estimate of a low dimensional and of the above system is the following: For low dimensional systems, we saw the laminar behavior on large length scales, while for the CML the laminar part appears on small length scales. This is due to the frequency the periodic parts appear. They are relatively seldom, which means that $p'_{\rm out}\gg p'_{\rm in}$, so that the coarse grained invariant measure of the system on large scales is dominated by the chaotic parts and the correlation dimension is roughly identical to $D_2^{\rm out}(\epsilon,m)$. The singularities [and thus $D_2^{\rm in}(\epsilon,m)$] only appear if we reduce the length scale further and further. From Fig. 8 we see that the higher the embedding dimension, the more abrupt is the breakdown. The reason for this behavior is that due to the high dimensionality of the attractor, all center points in the chaotic parts lose their neighbors on large length scales, so that finally the only pairs contributing to the correlation sum stem from the laminar phase. This holds for arbitrarily high embedding dimension, since the temporal length of the laminar phases $\langle l \rangle$ is not bounded. Of course, this loss of neighbors is faster the higher the embedding dimension is (at least as long as the embedding dimension is smaller the attractor's dimension, which is the case here).

To further illustrate that the breakdown is indeed caused by the laminar parts we checked it directly in the following way: When computing the correlation integral we excluded all points as center points which lay in the laminar parts. That is, we only computed $D_2^{\mathrm{out}}(\epsilon,m)$. The result is shown in Fig. 9. There one clearly sees that the breakdown does no longer appear and that the dimension grows more or less linearly with $\ln(\epsilon)$.

We showed that the existence of laminar parts in a system can produce spurious dimension estimates for low as well as for high dimensional (spatially extended) systems. While this effect may not be too severe for low dimensional systems, since we may have enough data to resolve the intermittent structure, for high dimensional systems these effects could pretend a dimensionality of the system, which cannot be resolved by going to smaller length scales. This becomes clear if we look at Fig. 7 where the breakdown is not as pronounced as in Fig. 8. The ''breakdown'' in the former figure could also be interpreted as a plateau, which would lead to a dimension estimate of about 4.5. This means, one has to be very careful in interpreting the results of the correlation dimension estimate, especially in a spatially extended system, where space-time intermittency seems to be a ubiquitous phenomenon.

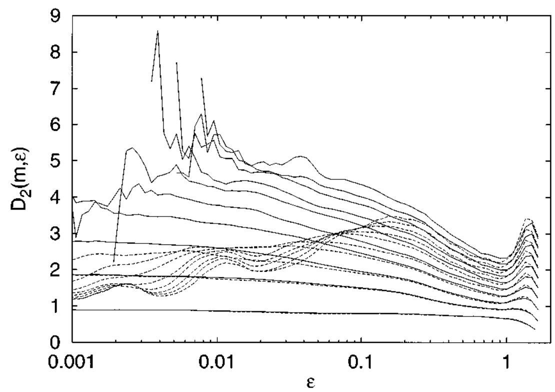

FIG. 9. Shown are the data from Fig. 8 and the estimate of $D_2^{\mathrm{out}}(\epsilon,m)$ $(m=1,\ldots,10)$ only, for the same system.

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